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Asymptotic and Essentially Singular Solutions of the Feigenbaum Equation

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For suitably defined large N, we express Feigenbaum's equation as a singular Schroder functional equation whose solution is obtained using a scaling ansatz. In the limit of infinite N certain self-consistency conditions on the scaled Schroder solution lead to an essentially singular solution of Feigenbaum's equation with a length scale factor of $\alpha \simeq 0.0333$ and a limiting feigenvalue of $\delta_\infty \simeq 30.50$, in agreement with Eckmann and Wittwer's value of $\alpha = 0.0333831...$ and their conjectured estimate of $\delta_\infty \lesssim 30$.

KEY WORDS: Feigenbaum; large N; iteration; doubling transformation; universal; fixed point; feigenvalue.

1. INTRODUCTION

Feigenbaum's original discovery[1] of universal properties of maps on an interval stimulated interest in the functional equation

$$\mathbf{T} \cdot f(x) \equiv \alpha^{-1} f(f(\alpha x)) = f(x) \tag{1}$$

This equation is sometimes referred to as the Cvitanović–Feigenbaum (CF) equation, from which certain universal constants can be computed.

Solutions of the CF equation can be thought of as fixed points of the doubling transformation $\mathbf{T}$ defined in Eq. (1) and acting on some appropriate space of functions. A convenient function space to consider[2] are maps f on the unit interval $[0, 1]$ which decrease monotonically from

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$f(0)=1$ to a minimum at some point $\lambda^{-1} \in (0, 1)$ and then increase monotonically on $[\lambda^{-1}, 1]$ to $f(1)=\alpha$. Such functions, often called unimodular, are equivalent under topological conjugacy[2] to classes of functions considered by other authors.[3]

The action of $\mathbf{T}$ on unimodular functions having the form

$$f_N(x; \lambda) \sim (1-\lambda x)^{2N} \qquad \text{as} \quad x \to \lambda^{-1} \tag{2}$$

in the neighborhood of their minimum is of particular interest. Iterates of such functions are known to undergo bifurcations through a cascade of 2^k-cycles at values $\lambda = \lambda_k$ which typically converge monotonically to a critical value λ_c.[4] The rate of convergence

$$\lambda_c - \lambda_k \sim \delta_N^{-k} \tag{3}$$

is exponential and is governed by a universal constant δ_N.[1] This constant depends only on N, but with the proviso that $f'_N(0; \lambda)$ is strictly negative.[2] Typical values of δ_N are given in Table II.

Fixed point functions f_N^* of $\mathbf{T}$ having the form (2) in the neighborhood of their minima are known to exist[5-9] and it is commonly believed that if λ is set equal to its appropriate critical value in any unimodular function f_N, $\mathbf{T}^n \circ f_N$ will converge to f_N^* and, moreover, that the feigenvalue δ_N is the maximum eigenvalue of the linearization of $\mathbf{T}$ around f_N^*.[1-3]

While certain aspects of the "renormalization group" scheme can be made rigorous,[5-9] it still remains to classify the universality classes of functions having the same δ_N, or equivalently, to classify the basins of attraction of fixed point functions of $\mathbf{T}$. Topological conjugates of the f_N^*, at least for unimodular and perhaps analytic f_N^*, seem to be prime candidates for universality classes,[2] but as yet there is no proof.

The existence and asymptotic behavior of f_N^*, and in particular their associated universal constants δ_N and $\alpha_N = f_N^*(1)$ as $N \to \infty$, are of some interest and have been the subject of study by several authors.[8,10-14] Eckmann and Wittwer,[8] for example, have written a whole book on this question. They stress the possible relevance of this problem to other "large-N" problems in physics.

From a numerical point of view the large-N problem for the CF equation is extremely delicate. The windows of stability of successive 2^k-cycles become prohibitively small, at least from a computational point of view, even for moderate N (of about 5 or so). In fact, it is not difficult to convince oneself from numerical evidence that δ_N diverges and α_N converges to zero as $N \to \infty$. In their computer-aided proof, however, of the existence of a limiting fixed-point function of a functional transformation derived from (1), Eckmann and Wittwer[8] assert that δ_N is bounded above by about 30 and that $\alpha_N \to 0.03338...$ as $N \to \infty$. This assertion is in conflict with recent work by Groeneveld,[13] who obtained upper bounds on α_N that converge to zero for particular nonanalytic solutions of the CF equation.

Our aim here is to reexamine the large-N problem in the light of this conflict and in particular to study asymptotic and essentially singular solutions of (1) in the limit $N \to \infty$. We work directly with the CF equation rather than with the derived functional equations and transformations considered by Eckmann and Wittwer.[8]

In the following section we transform the CF equation into a Schroder functional equation,[15] which becomes singular in the limit $N \to \infty$. A scaling ansatz is used to obtain asymptotic approximations to solutions of the singular Schroder equation.

The scaled asymptotic solutions to the Schroder equation are then used in Section 3 to obtain asymptotic solutions to the CF equation for large N. In the limit $N \to \infty$ these solutions lead to a unimodular solution of the CF equation that has an essential singularity at its minimum. We find that the scaling ansatz and certain assumptions of analyticity imply that α_N must approach a nonzero limit as $N \to \infty$. There is no contradiction with Groeneveld's results,[13] however, since his particular solutions are nonanalytic.

Numerical results presented in Section 4 agree with those obtained by Eckmann and Wittwer.[8] Our previously published algorithm for computing the feigenvalue $\delta^{(2)}$ from a fixed point function is used to obtain $\delta_\infty \simeq 30.50$ for our solution, in agreement with Eckmann and Wittwer's conjecture of $\delta_\infty \lesssim 30$. Our conclusions are summarized in the final section.

2. SCALED ASYMPTOTIC SOLUTION OF A SINGULAR SCHRODER EQUATION

In order to study asymptotic unimodular solutions of (1) that have the form (2) in the neighborhood of their minimum, we write

$$f(x) = [S_N(x)]^{2N}, \qquad f(1) = \alpha_N < 1 \tag{4}$$

in Eq. (1) and assume for simplicity that N is a positive integer. We can then assume without loss of generality that S_N is strictly monotone decreasing on $[0, 1]$, vanishes at some point $b_N \in (0, 1)$, and takes the values $S_N(0) = 1$ and

$$S_N(1) = -\exp\left(\frac{1}{2N} \log \alpha_N\right) \tag{5}$$

Substituting (4) into (1) and using (5), we easily obtain the Schroder functional equation for $S_N(x)$,

$$S_N(\varphi_N(x)) = (-1 + \varepsilon_N^2)\, S_N(x) \tag{6}$$

where

$$\varepsilon_N^2 = 1 - \exp\left(\frac{1}{2N}\log \alpha_N\right) > 0 \tag{7}$$

and

$$\varphi_N(x) = [S_N(\alpha_N x)]^{2N} \tag{8}$$

Disregarding Eq. (8) and subscripts N for the moment, the problem posed by the Schroder equation (6) is: given ε and a decreasing analytic function φ on [0, 1] with $\varphi(0) = 1$, find a decreasing analytic function S on [0, 1] such that

$$S(\varphi(x)) = (-1 + \varepsilon^2)\, S(x), \qquad S(0) = 1 \tag{9}$$

Granted the above conditions on φ and S and assuming $\varepsilon^2 < 1$, certain other conditions follow. Thus, if b is the unique fixed point of φ in [0, 1],

$$\varphi(b) = b, \qquad 0 < b < 1 \tag{10}$$

it follows on substituting $x = b$ in (9) that

$$S(b) = 0 \tag{11}$$

Also, if we differentiate (9) with respect to x and set $x = b$, it follows, since $S'(b)$ is finite and nonzero, that

$$\varphi'(b) = -1 + \varepsilon^2 \tag{12}$$

Setting $x = 0$ in (9) gives in addition the boundary condition

$$S(1) = -1 + \varepsilon^2 \tag{13}$$

There is an enormous literature on the above problem[15] and much is known for the case $\varepsilon \neq 0$. In our situation, however, ε given by (7) becomes arbitrarily small for large N (assuming here and henceforth that α_N does not approach zero exponentially fast as $N \to \infty$), so we are particularly interested in the asymptotic form of the solution as $\varepsilon \to 0+$. This is clearly a singular limit, since when $\varepsilon = 0$ the only solution to (9) satisfying the stated conditions is

$$S(x) = \begin{cases} 1 & \text{when} \quad 0 \leqslant x < b \\ 0 & \text{when} \quad x = b \\ -1 & \text{when} \quad b < x \leqslant 1 \end{cases} \tag{14}$$

which is singular at b.

As far as we are aware, the asymptotic behavior of solutions to the Schroder equation (9) for small ε has not been treated in the literature. Familiarity with similar singular problems in critical phenomena, however, suggests the scaling ansatz

$$S(x) = F(h(1 - x/b)/\varepsilon^{\Delta})/F(h(1)/\varepsilon^{\Delta}) \tag{15}$$

for solution as $\varepsilon \to 0+$ with Δ some scaling exponent and F and h analytic, monotone increasing, and satisfying

$$F(0) = h(0) = 0 \tag{16}$$

$$\lim_{x \to \pm\infty} F(x) = \pm 1 \tag{17}$$

Expanding φ around its fixed point b and noting conditions (10) and (12), we have

$$\varphi(x) = b + (\varepsilon^2 - 1)(x - b) + \tfrac{1}{2}\varphi''(b)(x - b)^2 + \tfrac{1}{6}\varphi'''(b)(x - b)^3 + \cdots \tag{18}$$

Also assuming h is analytic in some neighborhood of the origin, we can write, using (16),

$$h(x) = x + sx^2 + tx^3 + ux^4 + \cdots \tag{19}$$

Substituting (18) and (19) into (9) and (15) and matching terms requires a scaling exponent $\Delta = 1$. Moreover, from (18) and (19) we have

$$h(1 - \varphi(x)/b)/\varepsilon = y - A\varepsilon - B\varepsilon^2 - C\varepsilon^3 - \cdots \tag{20}$$

where in terms of the scaled variable $y = [(x/b) - 1]/\varepsilon$,

$$\begin{gathered} A = (a - s)\, y^2, \qquad B = y + (c + 2as - t)\, y^3 \\ C = y^2[2s + y^2(2cs + 3at - a^2 s - u + d)] \end{gathered} \tag{21}$$

and

$$a = \frac{b}{2}\,\varphi''(b), \qquad c = \frac{b^2}{6}\,\varphi'''(b), \qquad d = \frac{b^3}{24}\,\varphi''''(b), \ldots \tag{22}$$

After some straightforward but tedious algebra we then obtain

$$\begin{aligned} &F(h(1)/\varepsilon)[S(\varphi(x))+(1-\varepsilon^2)\,S(x)] \\ &\quad=[F(y)+F(-y)]-\varepsilon[AF'(y)-sy^2F'(-y)] \\ &\qquad+\varepsilon^2[\tfrac{1}{2}A^2F''(y)-BF'(y)+\tfrac{1}{2}s^2y^4F''(-y)-ty^3F'(-y)-F(-y)] \\ &\qquad+O(\varepsilon^3) \end{aligned} \tag{23}$$

Equating coefficients of ε^0, ε, and ε^2 to zero gives, respectively,

$$F(y)=-F(-y) \tag{24}$$

$$s=a/2 \tag{25}$$

and

$$y[1+(c+a^2)\,y^2]\,F'(y)=F(y) \tag{26}$$

i.e.,

$$F(y)=Dy[1+(c+a^2)\,y^2]^{-1/2} \tag{27}$$

where, in deriving (25) and (26), use has been made of (24), and of (24) and (25), respectively, and the constant D is arbitrary.

It will be noted from (27) that (24) is satisfied *a fortiori* and that (17) is also satisfied for the choice $D=(c+a^2)^{1/2}$.

Equation (23) then shows that with the choice $s=a/2$ and $F(y)$ given by (27), the function

$$S^0(x)=Dh\left(1-\frac{x}{b}\right)\left\{\varepsilon^2+(c+a^2)\left[h\left(1-\frac{x}{b}\right)\right]^2\right\}^{-1/2} \tag{28}$$

with D such that $S^0(0)=1$, satisfies the Schroder equation (9) with an error of order ε^3. To this order we can reexpress (28) in the form

$$\begin{aligned} S^0(x)&=\operatorname{sign}\left(1-\frac{x}{b}\right)\left\{1+\frac{\varepsilon^2}{(c+a^2)[h(1-x/b)]^2}\right\}^{-1/2}\left\{1+\frac{\varepsilon^2}{(c+a^2)[h(1)]^2}\right\}^{1/2} \\ &=\begin{cases}1-\varepsilon^2 f(x)+O(\varepsilon^4) & \text{when } 0\leqslant x<b \\ -1+\varepsilon^2 f(x)+O(\varepsilon^4) & \text{when } b<x\leqslant 1\end{cases} \end{aligned} \tag{29}$$

where

$$f(x)=\left\{\frac{1}{[h(1-x/b)]^2}-\frac{1}{[h(1)]^2}\right\}\frac{1}{2(c+a^2)} \tag{30}$$

The boundary condition (13) is then satisfied to leading order when

$$f(1)=1 \tag{31}$$

which may be viewed as an additional relation between the (unknown) coefficients t, u,... in the expansion (19) of $h(x)$ and (known) higher order derivatives of $\varphi(x)$ [Eq. (22)] at $x=b$. One could, for example, satisfy the Schroder equation to order ε^3 with a cubic form ($u=0$) for $h(x)$ by adjusting t to satisfy (31).

The above asymptotic approximation can be systematically improved, but there appears to be no simple algorithm for determining the coefficients of $h(x)$ from the derivatives of $\varphi(x)$ at $x=b$. To next order, for example, the ε^3 term on the right-hand side of (23) is easily found to be

$$\begin{aligned}\Delta = &-3a\,Dy^2\,\varepsilon^3\{1+\tfrac{1}{3}[5c+6t+2a^2+2(d-2u)/a]\,y^2\}\\ &\times\ \{2[1+(c+a^2)\,y^2]^{3/2}\}^{-1}\end{aligned} \tag{32}$$

In terms of the original variable $x=b(\varepsilon y+1)$ it is clear that this term is minimized and of order ε^4 when the coefficient of y^4 vanishes. That is, when

$$u=(d+a^3+3at+5ac/2)/2 \tag{33}$$

In this case

$$\Delta=-\frac{3}{2}a\varepsilon^4 D\left(\frac{x}{b}-1\right)^2\left[\varepsilon^2+(c+a^2)\left(\frac{x}{b}-1\right)^2\right]^{-3/2} \tag{34}$$

and it easily follows that

$$S^1(x)=S^0(x)\left\{1-\frac{3}{4}a\varepsilon^4 D\left(\frac{x}{b}-1\right)\left[\varepsilon^2+(c+a^2)\left(\frac{x}{b}-1\right)^2\right]^{-3/2}\right\} \tag{35}$$

is an asymptotic approximation to the solution of the Schroder equation (9) with error of order ε^5. To this order we then have a quartic approximation to $h(x)$, Eq. (19), with u given by (33), s by (25), and t again determined by the boundary condition (13) or (31).

In the following section we use the above results to develop a large-N asymptotic solution to the CF equation.

3. LARGE-N ASYMPTOTIC AND ESSENTIALLY SINGULAR SOLUTIONS TO THE CF EQUATION

For large N the small parameter ε of the previous section is given, from (7), by

$$\begin{aligned}\varepsilon^2 &= 1-\exp\left(\frac{1}{2N}\log\alpha_N\right)\\ &\sim -\frac{1}{2N}\log\alpha_N\equiv\varepsilon_N^2\end{aligned} \tag{36}$$

and the large-N asymptotic solution of the CF equation is given, from (4) and (28) (after normalization to unity at the origin), by

$$\begin{aligned} f_N(x) &\sim [S^0(x)]^{2N} \\ &= \left\{\frac{\varepsilon_N^2 + (c+a^2)[h(1)]^2}{\varepsilon_N^2 + (c+a^2)[h(1-x/b)]^2}\right\}^N \left\{\frac{h(1-x/b)}{h(1)}\right\}^{2N} \end{aligned} \tag{37}$$

The various coefficients a, c, etc., which are expressed in terms of the derivatives of $\varphi_N(x)$ at its fixed point, must now be determined self-consistently by the requirement, given from (8) and (28), that

$$\varphi_N(x) \sim [S^0(\alpha_N x)]^{2N} \tag{38}$$

If we now assume, with Eckmann and Wittwer,[8] that $\alpha_N \to \alpha \neq 0$ and $\alpha < b$, we obtain from (29), (36), and (38) the asymptotic form

$$\varphi_N(x) \sim \varphi(x) \equiv \exp[\log \alpha f(\alpha x)] \tag{39}$$

where $f(x)$ is given by (30).

Furthermore, if we substitute the asymptotic form (29) into the Schroder equation (9) and note from our monotonicity assumption that $\varphi(x) \gtrless b$ when $x \lessgtr b$, we obtain, using (39), the functional equation

$$f(\varphi(x)) = 1 + f(x) + O(\varepsilon_N^2) \qquad \text{for} \quad 0 \leqslant x \leqslant b \tag{40}$$

If we now multiply (40) by $\log \alpha$ and make use again of (39), we recover in the limit $N \to \infty$ the CF equation

$$F(F(\alpha x)) = \alpha F(x) \tag{41}$$

where

$$F(x) = \exp[\log \alpha f(x)] \tag{42}$$

and from (30)

$$f(x) = \left\{\frac{1}{[h(1-x/b)]^2} - \frac{1}{[h(1)]^2}\right\} \frac{1}{2(c+a^2)} \tag{43}$$

Since our initial ansatz assumes that $h(x)$ is analytic and monotonic increasing with $h(0) = 0$, it follows from (41)–(43) that our asymptotic solution of the Schroder equation yields a solution of the CF equation that has an essential singularity at its minimum $x = b$, i.e.,

$$F^{(n)}(b) = 0 \qquad \text{for all} \quad n = 0, 1, 2, \ldots \tag{44}$$

The question of the existence of a finite limiting value of α, granted our assumptions of analyticity, therefore depends on the existence of an essentially singular solution to the original CF equation. In the next section we provide numerical evidence for the existence of such a solution. Since our computed value of α turns out to be very close to the value obtained by Eckmann and Wittwer,[8] it may in fact be possible to interpret their results as a computer-aided "proof" of the existence of an essentially singular solution to the CF equation.

Finally, if we retain our assumption of analyticity and assume that in fact $\alpha_N \to 0$ as $N \to \infty$, we obtain from (8) the result

$$\begin{aligned}\varphi(x) &= \lim_{N \to \infty} (1 - \alpha_N \, |S'_N(0)| \, x)^{2N} \\ &= \exp(-Kx) \qquad (45)\end{aligned}$$

provided

$$K = \lim_{N \to \infty} 2N\alpha_N \, |S'_N(0)| \neq 0 \quad \text{exists} \qquad (46)$$

However, for small x we have from (29) and (30) that

$$S(x) \sim 1 - \varepsilon_N^2 x h'(1)/(c + a^2)[h(1)]^3 \qquad (47)$$

and hence from (36) that

$$2N\alpha_N \, |S'_N(0)| \sim (\alpha_N \log \alpha_N) \, h'(1)/(c + a^2)[h(1)]^3 \qquad (48)$$

It follows from (46) and (48) that if $\alpha_N \to 0$, $K = 0$, so we reach a contradiction. We conclude that h cannot be analytic and hence that there can be no analytic solution of the CF equation that has $\alpha_N \to 0+$. There is no contradiction with Groeneveld's results,[13] however, since his class of solutions are nonanalytic.

4. NUMERICAL RESULTS

4.1. The Fixed Point Function

In the preceding section we showed that the essentially singular function F given by (42) is a solution of the CF equation so long as

$$\varphi(x) = F(\alpha x) = \exp[\log \alpha f(\alpha x)] \qquad (49)$$

with f given by (43), satisfies certain conditions. For example, $\varphi(b) = b$

[Eq. (10)] and $\varphi'(b) = -1$ [Eq. (12)] (in the limit $N \to \infty$) require, respectively,

$$b = \exp[\log \alpha f(\alpha b)] \tag{50}$$

and

$$\alpha = -[b \log \alpha f'(\alpha b)]^{-1} \tag{51}$$

In addition, the function f is given in terms of h, Eq. (43), whose coefficients in turn must be determined self-consistently from φ through Eqs. (22), (25), (31), and (33). That is,

$$f(x) = \left\{\frac{1}{[h(1-x/b)]^2} - \frac{1}{[h(1)]^2}\right\} \frac{1}{2(c+a^2)} \tag{52}$$

where

$$h(x) = x + \tfrac{1}{2}ax^2 + tx^3 + ux^4 \tag{53}$$

$$\begin{aligned} a &= \tfrac{1}{2}b\varphi''(b) \\ &= \tfrac{1}{2}(1 - 3\alpha h'/h + ah''/h') \end{aligned} \tag{54}$$

$$\begin{aligned} c &= \tfrac{1}{6}b^2\varphi'''(b) \\ &= \tfrac{1}{6}\{-1 + 3(1-2a) - \alpha^2[h'''/h' - 9h''/h + 12(h'/h)^2]\} \end{aligned} \tag{55}$$

etc., with $h(x)$ and its derivatives $h'(x)$... evaluated at $x = 1 - \alpha$, u is given by (33), and t is to be determined from the boundary condition (31), i.e., from (42)

$$F(1) = \alpha \tag{56}$$

It is also not difficult to show from (56), assuming h is analytic and monotonic, that there is a unique "solution" for α in the interval (0, 1).

If we now start with the initial "guess" $h(x) = x$ we obtain, from (54) and (55),

$$a = \tfrac{1}{2}(1-4\alpha)(1-\alpha)^{-1}, \qquad 2(c+a^2) = \tfrac{1}{6} + \tfrac{1}{2}[\alpha/(1-\alpha)]^2 \tag{57}$$

and then from (50) and (51) the equations

$$b = \exp[-\tfrac{1}{2}(1-\alpha)(2-\alpha)] \tag{58}$$

and

$$\alpha \log \alpha = -\tfrac{1}{12}(1-\alpha)(1-2\alpha+4\alpha^2) \tag{59}$$

respectively, which have the solution

$$\alpha = 0.020, \qquad b = 0.379 \tag{60}$$

Using the above as a guide, we next choose trial α, b, a, t, and u, compute a and c from (54) and (55), and readjust t to satisfy the boundary condition (56). We found using this procedure that $a \approx 0.44$ and $2(c + a^2) \approx 0.168$ are relatively insensitive to trial α and b, which were subsequently recomputed using the trial values of α and b in the right-hand side of (51) and (50), respectively. We also found by trial and error that to quartic order, a better fixed-point function solution to (41), of the form (42) and (43), which is accurate to within a few percent over the entire interval $[0, 1]$, is obtained by "fine-tuning" t and u to satisfy (56) and the "second-order boundary condition"

$$F(F(\alpha)) = \alpha^2 \tag{61}$$

obtained from (41) by substituting $x = 1$.

The final results of our numerical analysis of the essentially singular solution to the CF equation are:

$$\alpha = 0.0333..., \qquad b = 0.3912... \tag{62}$$

with auxiliary parameters

$$\begin{aligned} a &= 0.435490, \qquad 2(c + a^2) = 0.168593 \\ t &= 0.2708, \qquad u = 0.12054 \end{aligned} \tag{63}$$

The results for α and b in particular are in striking agreement with the Eckmann–Wittwer values[8] of $\alpha = 0.033381...$ and $b = 0.391133...$.

4.2. Computation of δ

In a previous publication[2] we derived an algorithm for the computation of δ. In this algorithm the lth approximant to δ is given by

$$\delta^{(l)} = X_{l+1}/X_l, \qquad l = 0, 1,... \tag{64}$$

where $X_0 \equiv 1/f'(0)$, and

$$X_l = \alpha^{-l} \sum_{k=1}^{2^l - 1} f^{(k)}(0)/f^{(k)\prime}(0) \tag{65}$$

where f is a fixed point of the doubling transformation $\mathbf{T}$, i.e., some solution to the CF equation, $f^{(k)}$ denotes the kth iterate of f, and $f^{(k)\prime}$ is its derivative. The algorithm in fact simply provides an efficient way of computing the largest eigenvalue of the linearization of $\mathbf{T}$ around f. In the traditional parabolic case, $N=1$ in Eq. (2), for example, we obtain

$$\delta^{(1)}=4.671, \qquad \delta^{(2)}=4.66922, \qquad \delta^{(3)}=4.669204, \qquad \delta^{(4)}=4.6692019,... \tag{66}$$

which is to be compared with the "exact value" of $\delta_1=4.6692016...$.

For an "exact" fixed point function the $\delta^{(l)}$ actually converge to the appropriate δ. In practice, however, where f is known only approximately, the $\delta^{(l)}$ typically behave like a standard asymptotic sequence, at first converging, but eventually diverging, the point of divergence depending on how accurately f is known.

For large N the task of computing δ_N from numerical data obtained from bifurcating 2^k-cycles becomes almost impossible. Moreover, in the limit $N\to\infty$ where we have to deal with functions having an essential singularity at their minimum it is virtually impossible to detect anything beyond a 4-cycle. In this situation one could even question the existence of the bifurcating 2^k-cycle pattern. Our preference is to view the feigenvalue δ as a "divergence parameter" computed as the maximum eigenvalue of a linearized doubling transformation.

For the approximate fixed point function F obtained above, the algorithm (64), (65) gives the asymptotic sequence $\delta_\infty^{(l)}$ for $\delta_\infty=\lim_{N\to\infty}\delta_N$:

$$\delta_\infty^{(l)}=31.06, 30.40, 30.52, 30.97, 30.91, 30.68, 2608.86,... \quad \text{for} \quad l=1, 2,..., 7, \text{ respectively} \tag{67}$$

The divergence, it will be noted, is quite spectacular, but on the basis of previous experience with divergent asymptotic sequences we feel confident in predicting the estimate

$$\delta_\infty=30.50\pm 0.6 \tag{68}$$

Table I. Values for $a_N=(\alpha_N)^{-1/2N}$ Using Asymptotic Estimates for α_N and Other Numerical Methods

N	Asymptotic estimate	Numerical estimate[12–14]
5	1.33	1.29
50	1.034	1.0337
100	1.0170	1.0173
250	1.0068	1.0071
500	1.0034	1.0035

This is certainly in accord with Eckmann and Wittwer's conjecture[8] of $\delta_\infty \lesssim 30$.

Further comments and details on the algorithm (64), (65) are given in the Appendix.

4.3. Large-N Estimates

For large N the asymptotic form of the fixed point function $f_N(x)$ is given, to leading order, by Eq. (37), where from (36) and the boundary condition (13),

$$\begin{aligned} f_N(1) &= [S_N(1)]^{2N} \\ &\sim \left(1 + \frac{1}{2N} \log \alpha_N\right)^{2N} \sim \alpha \qquad \text{as} \quad N \to \infty \end{aligned} \tag{69}$$

That is,

$$\alpha_N \sim \exp[2N(\alpha^{1/2N} - 1)] \qquad \text{as} \quad N \to \infty \tag{70}$$

Values for α_N obtained from (70) and the estimate $\alpha = 0.0333$ are given in Table I. For comparison we include numerical estimates obtained independently by other authors, where their scale factor $a_N = (\alpha_N)^{-1/2N}$ is related to ours through a topological conjugacy.[2]

We have also used the asymptotic form (37) and the algorithm (64), (65) to compute estimates for δ_N based on the numerical results given in Eqs. (62) and (63). The results for the first two asymptotic estimates obtained from (64) and (65) are given in Table II.

Table II. Asymptotic Estimates for δ_N and Some Values Computed Numerically

	δ_N		
N	Ref. 1	Ref. 2	Numerical value[12]
1	5.39	4.53	4.67
2	10.14	9.17	6.08
5	18.24	17.22	12.30
50	29.01	27.98	—
100	29.86	28.83	—
250	30.38	29.27	—
500	30.56	29.54	—
∞	31.06	30.40	—

5. SUMMARY AND CONCLUSIONS

In this paper we have investigated large-N asymptotic and essentially singular solutions to Feigenbaum's functional equation.

For large N the problem can be expressed in terms of a singular Schroder functional equation whose general asymptotic solution was obtained using a scaling ansatz and certain monotonicity and analyticity assumptions. In the limit of infinite N we showed that certain self-consistency conditions on the Schroder asymptotic solution led to an essentially singular solution of the CF equation with a length scale factor of $\alpha \approx 0.0333$.

Our results are in accord with computer-aided proof results of Eckmann and Wittwer,[8] but in apparent conflict with recent results of Groeneveld,[13] who constructed particular solutions to the CF equation having length scale factors converging to zero. There is no contradiction, however, since our sequence of functions and those of Eckmann and Wittwer are analytic, whereas those of Groeneveld are nonanalytic.

Parameter values for the infinite-N, essentially singular solution were used to obtain asymptotic solutions for large N from which asymptotic estimates for the length scale factors α_N and the feigenvalue δ_N were obtained.

Our algorithm for computing δ_N from the associated fixed point function is to be viewed as providing an asymptotic sequence of estimates for the largest eigenvalue of a linearized doubling transformation. In the limit $N \to \infty$ we find $\delta_N \sim \delta_\infty \approx 30.50 \pm 0.6$, which agrees with the conjecture of Eckmann and Wittwer.[8]

It is to be stressed that there are severe numerical difficulties in computing δ_N from the occurrence of bifurcating 2^k-cycles for large N. In the limiting case of functions with an essential singularity it is virtually impossible to distinguish anything beyond a 4-cycle.

The question of universality classes, that is, the problem of classifying classes of functions that are attracted to particular solutions of the CF equation under the action of the doubling transformation (1), remains an interesting and open question. In this respect our essentially singular solution provides a particularly challenging problem.

APPENDIX. THE δ ALGORITHM

The algorithm (64) for computing δ involves iterates $f^{(k)}$ of the fixed point function f and their derivatives $f^{(k)\prime}$ evaluated at the origin. Specifically the lth approximant to δ is given by

$$\delta^{(l)} = X_{l+1}/X_l \tag{A1}$$

where

$$X_0 = 1/f'(0)$$

and

$$X_l = \alpha^{-l} \sum_{k=1}^{2^l - 1} f^{(k)}(0)/f^{(k)\prime}(0), \qquad l = 1, 2,... \tag{A2}$$

The iterates $f^{(k)}$ are defined recursively by

$$f^{(k+1)}(x) = f^{(k)}(f(x)), \qquad f^{(0)}(x) = x \tag{A3}$$

where f is a fixed point of $\mathbf{T}$, i.e., a solution of the Feigenbaum equation

$$f(f(\alpha x)) = \alpha f(x), \qquad f(0) = 1 \tag{A4}$$

Differentiating (A3) and (A4) with respect to x, we have that

$$f^{(k+1)\prime}(x) = f^{(k)\prime}(f(x))\, f'(x) \tag{A5}$$

and

$$f'(f(\alpha x))\, f'(\alpha x) = f'(x) \tag{A6}$$

Setting $x = 0$ in (A6), for example, and assuming $f'(0) \neq 0$, we deduce that $f'(1) = 1$. Setting $x = 1$ in (A6) then gives

$$f'(f(\alpha))\, f'(\alpha) = 1 \tag{A7}$$

Similarly, if we set $x = 0$ and $x = 1$ in (A4) we have

$$f(1) = \alpha, \qquad f(f(\alpha)) = \alpha^2 \tag{A8}$$

and so forth. Repeated use of (A4) and (A6) and (A5) allows us to simplify (A2) considerably. Using the above equations, we obtain, for example,

$$\begin{aligned}
\delta^{(0)} &= \frac{1}{\alpha} \\
\delta^{(1)} &= \frac{1}{\alpha}\left(1 + \alpha + \frac{f(\alpha)}{f'(\alpha)}\right) \\
\delta^{(2)} &= \frac{1}{\alpha}\left[1 + \left(\alpha^2 + \frac{f(\alpha^2)}{f'(\alpha^2)} + \alpha\frac{f(\alpha)}{f'(\alpha)} + \frac{f(\alpha f(\alpha))}{f'(\alpha)\, f'(\alpha f(\alpha))}\right)\left(1 + \alpha + \frac{f(\alpha)}{f'(\alpha)}\right)^{-1}\right]
\end{aligned} \tag{A9}$$

In general we have that

$$\delta^{(l)} = \frac{1}{\alpha}\left(1 + \frac{Y_l}{Z_l}\right) \tag{A10}$$

where

$$Z_{l+1} = Z_l + Y_l, \qquad Z_1 = 1 \tag{A11}$$

Y_l is a sum of 2^l terms whose numerators are obtained from strings of $l\alpha$'s with either the identity function or f between successive α's. The corresponding denominator for each term is obtained by taking the product of derivatives of f, one for each f appearing in the numerator, and having the same argument. This construction is clearly seen in (A9) for $l = 2$. As a further illustration, when $l = 3$, the terms in Y_3 involving two f's and one identity are

$$\begin{gathered} f(\alpha f(\alpha^2))/f'(\alpha f(\alpha^2))\, f'(\alpha^2) \\ f(\alpha^2 f(\alpha))/f'(\alpha^2 f(\alpha))\, f'(\alpha) \\ \alpha f(\alpha f(\alpha))/f'(\alpha f(\alpha))\, f'(\alpha) \end{gathered} \tag{A12}$$

The term involving three f's is

$$f(\alpha f(\alpha f(\alpha)))/f'(\alpha f(\alpha f(\alpha)))\, f'(\alpha f(\alpha))\, f'(\alpha)$$

while the term involving no f's is simply α^3, and so on.

Expressed in this form, the algorithm can be easily programmed for a computer. As mentioned previously, however, the accuracy of the fixed point function imposes limits on how many times the algorithm can be applied.

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